CHARACTERIZATION OF THE FORMATION OF FILTER PAPER USING THE BARTLETT SPECTRUM OF THE FIBER STRUCTURE

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ABSTRACT

The formation index of filter paper is one of the most important characteristics used in industrial quality control. Its estimation is often based on subjective comparison chart rating or, even more objective, on the power spectrum of the paper structure observed on a transmission light table. It is shown that paper formation can be modeled as Gaussian random fields with a well defined class of correlation functions, and a formation index can be derived from the density of the Bartlett spectrum estimated from image data. More precisely, the formation index is the mean of the Bessel transform of the correlation taken for wavelengths between 2 and 5 mm.

Keywords: Image analysis, fiber systems, filter paper, formation, chart cloudiness, second order characteristics, Bartlett spectrum.

INTRODUCTION

Filter papers are used in a wide variety of fields, ranging from air to oil filters, see Durst et al. (2007). They consist of bounded fibers which are more or less randomly distributed. Except the specific paper weight (i.e. the weight per unit area, also called the nominal grammage), the weight distribution is a very important characteristic of paper. It influences many properties of filter papers such as flow rate, particle collection, efficiency, wet strength, porosity and dust holding capability. Thus, the characterization of the weight distribution is important for industrial quality control as well as for the development of new filter materials and technologies of manufacture.

It is easy to get an impression of the weight distribution when holding a sheet of paper up against light and observing the distribution of the optical density, known as the paper formation, chart cloudiness or flocculation. Assuming a constant absorption coefficient for the solid constituents of the paper structure, the local intensity of the transmitted light can be related to the local weight density by Lambert-Beer’s law. As a consequence there is a close relationship between weight distribution and formation and, in fact, often one does not distinguish between both. See Van den Akker (1949); McDonald et al. (1986); Lien and Liu (2006) for the computation of the grammage from the absorption of visible light. The use of soft X-radiation is suggested in Farrington (1988), and the influence of the choice of radiation on the transmittance is investigated for nonwoven fabrics in Boeckerman (1992) and for paper in Norman and Wahren (1976); Bergeron et al. (1988).

Usually, the formation is experimentally determined based on two-dimensional (2D) images of the paper structure. A transmission light table is used in order to ensure a homogeneous illumination and the images are acquired by a CCD-camera having a linear transfer function, such that the pixel values can be assumed to be approximately proportional to the corresponding local intensities. In the simplest case, a paper structure inspection can be based on subjective comparison chart rating, supported by an industrial standard consisting on well-formulated rules for image acquisition and rating. Nevertheless, the valid industrial norm on paper, board, pulps and related terms gives only a rough description of the terms 'formation' (manner in which the fibers are distributed, disposed and intermixed to constitute the paper) and 'lock-through' (structural appearance of a sheet of paper observed in diffuse transmitted light), see ISO 4046(E/F) (2012). Inspection systems based on image analysis include the computation of a (more or less objective) value for the 'formation index' from the image data. One should keep in mind that the formation is independent of the nominal grammage as well as on the variance of the local paper weight.
But what is exactly meant by the ”formation index”, and it is sufficient to characterize paper formation by only one number?

There is a huge number of publications on the characterization of paper formation, see e.g. Kallmes (1984); Cresson (1988); Cherkassky (1999); Drouin et al. (2001), see also Waterhouse et al. (1991); Praast and Göttscing (1991) for a very good survey on literature from the late 1980th and Chinga-Carrasco (2009) for newer developments. An intuitive characteristic for the formation is the mean paper flock size, going back to Robertson (1956), but until now there is no convincing method for segmenting flocks in gray-tone images. More useful methods are based on measuring the variance of the pixel values or, more general, the co-occurrence matrix of the image data, see Yuhara et al. (1986); Cresson (1988); Cresson and Luner (1990a;b). The approach presented in Pourdeyhimi and Kohel (2002) is motivated by a Poisson statistics for the centers of paper flocks (objects). On the one hand, these centers cannot safely be detected and, on the other hand, the computation of the ‘uniformity intex’ of the paper formation is based on the variation of the area fraction in a binarized image but even not on the flock centers.

Since woven textiles have a (more or less) periodic pattern, it seems to be obvious to apply Fourier methods for quality inspection, see e.g. Wang et al. (2011) and references therein, where slight deviations from the periodicity are detected based on the the correlation function of the pattern or, analogously, its counterpart in the inverse space – the so-called power spectrum. In Sara (1978); Norman (1986); Cresson (1988); Provatas et al. (1996); Cherkassky (1998); Lien and Liu (2006), the correlation function and the power spectrum are also suggested as characteristics for cloudiness of (non-periodic but macroscopically homogeneous) nonwovens and paper formation, respectively, see also Section 2.2 in Alava and Niskanen (2006). Sometimes the range of interaction , i.e. integral of the covariance function (also known as the integral range), is used as an formation index. Instead of a Fourier transform, Scharcanski (2006) uses a wavelet transform to extract a spectral density from the sheet formation.

Mathematical modeling of paper structure on a mesoscale can lead to a deeper understanding e.g. of the phenomenon of formation, see Cresson (1988); Cherkassky (1998); Antoine (2000); Gregersen and Niskanen (2000); Provatas et al. (2000); Sampson (2009), where the model parameters – so far they can easily be estimated from image data – serve as formation characteristics. Further approaches are based on modeling random structures by Markow Random Fields (MRF) and decomposing the image of the structure into ”different scales”, evaluating the degree of homogeneity on each scale and computing an overall degree of homogeneity, see Scholz and Claus (1999), who applied this approach originally on the structure of nonwovens (fleeces and felts), but in principle this works also for the evaluation of paper structures, where the degree of homogeneity can be seen as a formation index. Notice that the ”different scales” mentioned above are also known as the Laplacian pyramid of the image data, see Burt and Adelson (1983).

In the present article we use Gaussian Random Fields (GRFs) for modeling paper formation and, following the suggestion made in Xu (1996); Lien and Liu (2006), a Fourier approach is applied for computing a characteristic of paper formation. More precisely, we show that the formation of the investigated filter papers can be characterized by the density of the Bartlett spectrum, i.e. a spectral representation of the correlation function. Using a parametric approach for the Bartlett spectrum, we introduce one of the model parameters as an appropriate quantity indexing paper formation, and we apply the method of Koch et al. (2003) for an fast and unbiased estimation of the density of the Bartlett spectrum.

Figure 1: An image showing the formation of a filter paper (left) and a realization of a macroscopically homogeneous and isotropic GRF (right) with $k(x) = e^{-\lambda\|x\|}$ and $\lambda = 0.6$; the edge length of the images 102.4 mm.

MODELING PAPER FORMATION BY GAUSSIAN RANDOM FIELDS

A look on Figure 1 shows that the formation of filter paper is surely one of the most convincing applications for GRFs. The difference among the real structure on the left-hand side and the realization on the right-hand side, which is obtained from the adapted GRF, can be recognized only by experts. This is very important since, if a formation can really be modelled...
by a GRF, then the Bartlett spectrum of the GRF uniquely specifies formation.

Let a probability space \((\Omega, \mathcal{F}, P)\) be given. Then we denote by \(\Phi(\omega, x)\) a 2-dimensional, real-valued random function which, for every fixed \(x \in \mathbb{R}^2\) is a measurable function on \(\omega \in \Omega\). For simplifying the notation, the dependency on the underlying probability space will suppressed throughout the article, and thus we are setting \(\Phi_x = \Phi(\omega, x)\). The function \(\Phi_x\) is macroscopically homogeneous (i.e. stationary in the strict sense), if \(\Phi_x\) is invariant with respect to translations, i.e. if all its finite-dimensional distributions are independent of translations of \(\Phi_x\). Furthermore, \(\Phi_x\) is said to be isotropic, if it is invariant with respect to rotations around the origin. Finally, a random function \(\Phi_x\) is forming a Gaussian Random Field (GRF) if all its finite-dimensional distributions are multivariate normal distributions, see e.g. Adler (1981); Abrahamsen (1985); Adler and Taylor (2007) for sound introductions to GRFs.

Fig. 2. Realizations of macroscopically homogeneous and isotropic GFRs for constant \(\mu\) and \(\sigma^2\) and the exponential correlation function \(k(\lambda) = e^{-\lambda ||x||}\) with the parameter \(\lambda = 0.1 \text{mm}^{-1}\), \(\ldots\), \(\lambda = 0.4 \text{mm}^{-1}\), lexicographic order. The edge length of the images is 102.4 mm.

A macroscopically homogeneous GRF \(\Phi_x\) is uniquely specified by its expectation \(\mu = \mathbb{E}\Phi_x\), its variance \(\sigma^2 = \mathbb{E}\Phi_x^2 - \mu^2\), \(\sigma > 0\), and a positive definite covariance function

\[
\text{cov}(x) = \mathbb{E}\left((\Phi_x - \mu)(\Phi_{x+x} - \mu)\right), \quad x \in \mathbb{R}^2,
\]

which is independent of \(y \in \mathbb{R}^2\). The normalized function \(k(x) = \text{cov}(x)/\sigma^2\) is known as the (auto-)correlation function. The expectation \(\mu\) is also called the first order characteristic of \(\Phi_x\), while \(\sigma^2\) and \(k\) are second order characteristics. All higher order characteristics depend only on \(\mu\), \(\sigma\) and \(k\). This is an important result from the theory of random fields, see e.g. Abrahamsen (1985), which means that our attention can be payed exclusively on the first and second order characteristics and their estimation.

Examples of realizations of a class of GRFs \(\Phi_x^{(\lambda)}\) with an exponential correlation function \(k(\lambda) = e^{-\lambda ||x||}\), \(x \in \mathbb{R}^2\), are shown in Figure 2. It turns out that the distributional properties of \(\Phi_x^{(\lambda)}\) distinguish by the scaling parameter \(\lambda > 0\), i.e. it holds \(\Phi_x^{(\lambda)} = \Phi_{\lambda x}^{(1)}\), and realizations of \(\Phi_x^{(\lambda)}\) can be obtained from realizations of \(\Phi_x^{(1)}\) by scaling.

Assume now, that a GRF is well adapted to the image data of a paper structure, then the interpretation of \(\mu\), \(\sigma^2\) and \(k\) is as follows: The expectation \(\mu\) is the brightness, \(\sigma\) corresponds to the image dynamics, and \(k\) is the correlation function of the pixel values. Under some technical conditions (using of a CCD camera allowing photometric measurements, high gray-tone resolution, constant gain, etc.), assuming Lambert-Beer’s law for light absorption and knowing the initial light intensity, the nominal paper grammage and the weight variance can roughly be estimated from \(\mu\) and \(\sigma^2\), respectively. As a consequence, the correlation function \(k\) determines the paper formation uniquely. Figure 3 shows realizations of two GRFs with constant \(\mu\) and \(k\) but with different \(\sigma\). One feels subjectively, that the formation is the same in both images.

Fig. 3. Realizations of two macroscopically homogeneous and isotropic GFRs with constant \(\mu\) and \(k(x) = e^{-\lambda ||x||}\) with \(\lambda = 0.5 \text{mm}^{-1}\); left: small \(\sigma\), right: larger \(\sigma\). The edge length of the images is 102.4 mm.

Finally, we remark that in the isotropic case the correlation function \(k\) depends on only the radial coordinate \(r = ||x||\) of \(x\), i.e. there is a function \(k_1\) such that \(k(x) = k_1(||x||) = k_1(r)\). Clearly, the exponential correlation function mentioned above is isotropic.
THE SPECTRAL REPRESENTATION OF THE CORRELATION FUNCTION

First of all, we recall Bochner’s theorem which states that the covariance function $\sigma^2k$ of a macroscopically homogeneous random field $\Phi$ can be represented by a non-negative, bounded measure $\Gamma$ – the so-called spectral measure or the Bartlett spectrum of $\Phi$. A proof is given e.g. in Katznelson (2004), p. 170. This very important theoretical result is also useful in applications, since efficient Monte Carlo techniques of generating realizations of GRFs as well as fast algorithms for estimating the second order characteristics are based on their spectral representations. In this article we restrict ourselves on the particular cases in which the Bartlett spectrum $\Gamma$ has a density $\gamma_\nu$, which is also known as the spectral density of $\Phi$. Let $\hat{k}$ be the Fourier transform of $k$, then $\gamma_\nu = \sigma^2\hat{k}$. In the usual setting, $k$ and $\hat{k}$ are related to each other by

$$\hat{k}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} k(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}^2, \quad (1)$$

and vice versa

$$k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{k}(\xi) e^{i\xi x} d\xi, \quad x \in \mathbb{R}^2. \quad (2)$$

For short, we will make use the notations $F$ and $F^*$ for the Fourier transform and its co-transform, allowing to rewrite the above relationships as $\hat{k} = Fk$ and $k = F^*k$, respectively.

Again, we refer on the isotropic case where also $\hat{k}$ depends on the corresponding radial coordinate $\rho = \|\xi\|$ and $\hat{k}_1(\rho) = \hat{k}(\xi)$ is Fourier-Bessel transform of $k_1(r)$,

$$\hat{k}_1(\rho) = \frac{1}{2\pi} \int_0^\infty r k_1(r) J_0(\rho r) dr, \quad \rho \geq 0, \quad (3)$$

where $J_0$ is the Bessel function of the first kind and of order 0. It is well known that the Fourier-Bessel transform of the exponential correlation function is

$$\hat{k}_1(\rho) = \frac{\sqrt{2}}{\pi} \frac{\lambda}{(\lambda^2 + \rho^2)^{3/2}}, \quad \rho \geq 0. \quad (4)$$

To be more flexible in the modeling and the characterization of paper formation, we introduce a generalized version of the exponential correlation function, which depends on an additional parameter $\nu > 0$. The modified Bessel correlation function is defined as

$$k_1(r) = \frac{(\lambda r)^\nu}{2^{\nu-1} \Gamma(\nu)} \mathcal{K}_\nu(\lambda r), \quad r \geq 0, \quad (5)$$

where $\Gamma$ denotes Euler’s $\Gamma$-function and $\mathcal{K}_\nu$ is the modified Bessel function of second kind and of order $\nu$. Note that $k_1(r) \to 1$ as $r \to 0$, and choosing $\nu = 1/2$ gives the exponential correlation function. The spectral density of the modified Bessel correlation function is

$$\hat{k}_1(\rho) = \frac{\lambda^{2\nu}}{2^{\nu-1} \Gamma(\nu) (\lambda^2 + \rho^2)^{\nu+1}}, \quad \rho \geq 0, \quad (6)$$


A GEOMETRIC INTERPRETATION

Bochner’s theorem states that a spectral representation exists for every continuous, positive definite function, i.e. also for functions which are not necessarily covariance functions of GRFs. To give an example, we consider a macroscopically homogeneous and isotropic 2D Boolean model $\Xi$ with identically distributed and pairwise independent random segments. In fact, Boolean segment processes can serve as appropriate models for real fiber systems of paper where the fibers are ‘scattered independently and uniformly’ in the paper sheet, see e.g. Provatas et al. (2000) who suggested a Boolean model with segments of constant length for modeling of fiber deposition.

Fig. 4. A realization of the isotropic Boolean model with segments of exponentially distributed lengths, $1/\lambda = 2\,\text{mm}$, $N_\lambda = 10\,\text{mm}^{-2}$; the width of the image is 20 mm.

In the following we assume that the length of the segments is exponentially distributed with the parameter $\lambda$, see Fig. 4 for a realization. Then the Boolean model $\Xi$ is uniquely characterized by the parameter $\lambda$ of the exponential distribution and the
specific line length \( L_A \), i.e. the expectation of the total length of the segments per unit area. Notice that \( 1/\lambda \) is the mean fiber length, and \( N_A = \lambda L_A \) is the mean number of fibers per unit area. Then
\[
g(r) = 1 + \frac{\lambda}{N_A \pi r} e^{-\lambda r}, \quad r > 0
\]
is the so-called pair correlation function of \( \Xi \), defined as the density of the reduced second moment which is associated with random length measure \( L \) of \( \Xi \), see p. 186 in Stoyan et al. (1995) for a formula of the reduced second moment measure \( K \) of Boolean segment processes.

Let \( \kappa : \mathbb{R}^2 \mapsto \mathbb{R} \) be a non-negative and bounded kernel function with \( \int_{\mathbb{R}^2} \kappa(x) dx = 1 \). By \( \kappa^t(x) = \kappa(-x) \) we denote the reflection of \( \kappa \), and \( \kappa * f \) is the convolution of the two functions \( \kappa \) and \( f \). Furthermore, let \( g : \mathbb{R} \mapsto \mathbb{R}^2 \) be an arclength parametrization of a finite immersed curve \( \phi \) in \( \mathbb{R}^2 \), that is \( \phi = \{ g(s) : 0 \leq s \leq \ell \} \), where \( g \) is twice continuous differentiable and \( \ell \) is the curve length. Then the convolution \( \phi * \kappa \) may be defined as
\[
(\phi * \kappa)(x) = \int_0^\ell \kappa(x - g(s)) ds, \quad x \in \mathbb{R}^2.
\]
Then \( \Psi_A = (\Xi * \kappa)(x) \) is a macroscopically homogeneous random field with \( E\Psi_A = L_A \), but \( \Psi_A \) is not a GRF.

If we choose \( \kappa \) such that it decreases sufficiently fast as \( ||x|| \to \infty \), then from the central limit theorem (CLT) it follows that
\[
\Phi_A = \lim_{N_A \to \infty} \frac{1}{N_A} (\Xi * \kappa - L_A)(x), \quad x \in \mathbb{R}^2
\]
forms a GRF with \( \mu = 0 \) and the covariance function \( \text{cov}(x) = \lambda^2 \lambda^2 h(x) \) where \( h(x) = g(||x||) - 1 \).

Let \( \{ \kappa_\varepsilon \}_\varepsilon > 0 \) be a family of non-negative kernel functions of bounded support, \( \kappa_\varepsilon(x) = 0 \) for \( ||x|| \leq \varepsilon/4 \), then it follows that \( \text{cov}(x) \to \sigma^2 h(x) \) as \( \varepsilon \downarrow 0 \) for all \( x \in \mathbb{R}^2 \).

In the line with the above, we are setting \( h_1(r) = g(r) - 1 \) and call \( h_1 \) the correlation function of the Boolean model \( \Xi \). It holds that \( h_1(r) \to \infty \) as \( r \downarrow 0 \), and the corresponding covariance measure is positive definite, see Section 6.4 in Ohser and Schladitz (2009). This means that there exists a Bartlett spectrum of \( \Xi \). Moreover, the Bartlett spectrum has a density, i.e. the Bessel transform
\[
h_1(\rho) = \frac{1}{\pi N_A} \frac{\lambda}{\sqrt{\lambda^2 + \rho^2}}, \quad \rho \geq 0
\]
of \( h_1 \) exists, which is, up to a constant factor, the same as \( \hat{k}_1 \) given in (6) for \( \nu = -\frac{1}{2} \). This is surprising, since the curve shape of \( h_1 \) basically differs from that of \( k_1 \) given in (5).

In other words, the GRF \( \Phi_\varepsilon \) inherits the second order properties of the Boolean model \( \Xi \). This shows that there is a close relationship between ‘independent and uniform scattering’ of fibers in the plane (observable on a microscale) and paper formation (observable on a mesoscale) where the fiber mean length \( 1/\lambda \) corresponds to the formation index. Nevertheless, ‘independent and uniform scattering’ of fibers means that there is no tendency to form fiber clusters (flocks) induced e.g. by adhesion and, moreover, a significant formation is observable even if the fibers are ‘independently and uniformly scattered’, see Fig. 5 for an example.

**Fig. 5. A realization of a GRF based on an isotropic Boolean model with segments of exponentially distributed lengths, \( 1/\lambda = 2 \text{mm} \), \( N_A = 10 \text{mm}^{-2} \), where the smoothing kernel \( \kappa \) is the probability density of the isotropic 2D Gauss distribution with \( \sigma = 0.02 \text{mm} \). The width of the image is 102.4 mm.**

**MONTE CARLO SIMULATION**

We follow the spectral approach developed by Shinozuka and Jan (1972) and others, where realizations of a GRF are generated based on the following two steps:

1. Let \( u \) be a random number uniformly distributed on the interval \([0, 1]\), and let \( v \) be a random vector distributed with respect to the probability measure \( \Gamma_\Phi / 2\pi \) on \( \mathbb{R}^2 \). If \( u \) and \( v \) are independent random variables, then
\[
\Psi_A = \sqrt{2} \cos(2\pi u + \pi x), \quad x \in \mathbb{R}^2
\]
forms a macroscopically homogeneous and isotropic random field with the expectation \( \mu = 0 \), the variance \( \sigma^2 = 1 \) and the correlation function \( k \). Notice that \( \Psi_1 \) is not ergodic.

2. Let now \( \psi_x^{(1)}, \ldots, \psi_x^{(m)}, m \in \mathbb{N} \), are pairwise independent and identically distributed random fields with \( \mu = 0, \sigma^2 = 1 \) and \( k \). Define 

\[
\phi_x^{(m)} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \psi_x^{(i)}, \quad x \in \mathbb{R}^2
\]

then from the CLT it follows that 

\[
\Phi_x = \sigma \lim_{m \to \infty} \phi_x^{(m)} + \mu, \quad x \in \mathbb{R}^2
\]

is a macroscopically homogeneous and isotropic GRF with \( \mu, \sigma^2 \) and \( k \).

See Lantuéjoul (2002) for further details and an overview of alternative approaches.

But how large must \( m \) be such that \( \phi_x^{(m)} \) can be seen as a realization of \( \Phi_x \)? The usual way for a suitable choice of \( m \) is based on the Berry-Esseen inequality

\[
\sup_x |\phi_x^{(m)} - \Phi_x| \leq \frac{c}{\sigma^2 \sqrt{m}}
\]

with \( c < 0.4784 \), see Korolev and Shevtsova (2010). For the realizations of the GRFs shown in Figures 1 to 3 the number \( m \) was empirically chosen as \( m = 4096 \), which surely is large enough as \( \hat{k}(\rho) \) vanishes rapidly at infinity, see also the remark in Lantuéjoul (2002), p. 192.

**ESTIMATION OF \( \hat{k} \)**

In this section we assume that the Bartlett spectrum \( \Gamma_{\Phi} \) of the observed random field \( \Phi_x \) has a density. We are starting from the normalized random field \( f(x) = (\Phi_x - \mu) / \sigma \) having the expectation 0 and the variance 1. In applications the function \( f \) is observed through a compact window \( W \subset \mathbb{R}^2 \) with the indicator function \( 1_W \) defined as \( 1_W(x) = 1 \) if \( x \in W \) and \( 1_W(x) = 0 \) otherwise. This means that the masked function \( f_W(x) = f(x) \cdot 1_W(x) \) is considered. One should keep in mind that the image data can be seen as a realization of \( f_W \), where \( W \) is the (rectangular) image frame. Furthermore, we introduce a window function \( c_W \) of \( W \) defined as the auto-correlation of the function \( 1_W \), \( c_W = 1_W \ast 1_W \).

The \( c_W \) is bounded and of bounded support, and thus its Fourier transform \( \hat{c}_W \) exists. This yields an analogue to the Wiener-Khintchine theorem, that is \( \mathcal{F}(c_W \cdot k) = 2\pi \mathbb{E} |\hat{f}_W| ^2 \). The power spectrum \( \mathbb{E} |\hat{f}_W| ^2 \) of \( f_W \) is integrable, and hence its inverse Fourier transform \( \mathcal{F} \) can be applied which yields 

\[
c_W \cdot k = 2\pi \mathcal{F} (\mathbb{E} |\hat{f}_W| ^2).
\]

Assume now that the origin belongs to \( W \). Then \( c_W \) is positive for all \( x \) belonging to the interior of \( W \), and it follows that 

\[
\frac{2\pi \mathcal{F}(|\hat{f}_W|^2)(x)}{c_W(x)}
\]

is an unbiased estimator of \( k(x) \) for all \( x \) in the interior of \( W \).

In the isotropic case the rotation mean of an estimation of \( k \) can be performed (rotation around the origin), which gives an estimation of the radial function \( k_1 \). From this, one obtains an estimation of the density \( \hat{k}_1 \) of the Bartlett-Spectrum using the 1-dimensional Fourier-Bessel transform (3).

An overview of the estimation procedure is given in Fig. 6. Clearly, \( k_{cw} \) can also be computed by correlation (red marked path in Fig. 6).

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**Fig. 6. Scheme of the computation of the density function \( \hat{k}_1(\rho) \). The symbol \( \ast^* \) stands for auto-correlation of functions (convolution with the reflected function).**

It is well known that for \( n \) pixels of an image of an image of \( \Phi_x \), the correlation function \( k \) can be computed using the Fast Fourier Transform (FFT) with a complexity in \( O(n \log n) \), where also the window function can efficiently be computed via the inverse space using \( c_W = \mathcal{F}[|\hat{1}_W|^2] \). In the rectangular case, the window function is explicitly known from the literature, see e.g. Ohser and Mücklich (2000), p. 356. A discrete version of the Bessel transform can be based on numerical integration, e.g. by Romberg’s rule.
Unfortunately, the assumption of periodicity in the discrete Fourier transform (dFT) causes an overlapping effect (edge effect). This effect can be eliminated by expanding the function \( f_w \) to the window \( 2W \), where \( f_w \) is padded with zeros, that is \( f_{2W}(x) = f_w(x) \) if \( x \in W \), and \( f_{2W}(x) = 0 \) if \( x \in 2W \setminus W \). This increases the pixel number to \( 4n \), still the complexity of the FFT applied to \( f_{2W} \) belongs to \( \mathcal{O}(n \log n) \) which is a considerable gain compared to the usual estimation of \( kc_w \) based on auto-correlation, see Fig. 6 (red path). Notice that data windowing using a 2D analogue of the Welch, Hann (Hamming) or Bartlett window, see Press et al. (2007), avoids any window expansion, but the unbiasedness of (7) gets lost.

The dFT (and its inverse) is usually based on a modified setting of the continuous Fourier transform. The main difference to be aware of, is that in (1) and (2) the frequency \( \xi \) is substituted with the angular frequency \( \omega = 2\pi\xi \). This has an impact on the scaling of the estimated spectral density.

Finally, we remark that the sampling of \( f \) on a homogeneous point lattice induces a sampling of \( \hat{f} \) on the inverse lattice, where the relationship between the original lattice and its inverse is as follows: Let \( U \) be a matrix for which the column vectors are forming a basis of the original lattice, then the column vectors of the matrix \( (U')^{-1} \) form a basis of the inverse lattice, Ohser and Schladitz (2009), p. 66. In terms of a dFT applied to a 2D image with \( n_1 \cdot n_2 \) pixels of size \( a_1 \cdot a_2 \), the transformed image also consists of \( n_1 \cdot n_2 \) pixels but of the size \( \hat{a}_1 \cdot \hat{a}_2 \), where \( \hat{a}_1 = 1/(n_1a_1) \) and \( \hat{a}_2 = 1/(n_2a_2) \).

**EXPERIMENTAL RESULTS**

The applicability of the method presented above is now demonstrated for three filter papers produced by wet laid cellulose fibers. The material No. 1 has a nominal grammage of 200 g/m² and a mean thickness of about 0.9 mm, the material No. 2 is of 90 g/m² and about 0.25 mm thick, and the material No. 3 is of 170 g/m² and about 0.7 mm thick. The mean fiber length in these materials was much longer than 2 mm.

In order to estimate the spectral density and to determine a formation index, various filter papers are scanned in the light transmission mode using a conventional CCD camera, see Figs. 7 to 9 (left) for examples. The 8-bit gray-tone images are of 1580 × 1200 pixels with a lateral resolution of 0.177 mm per pixel and, thus, the effective image size amounts 279.66 × 212.40 mm². The wet laying process induces a slight sheet inhomogeneity appearing as a long wave shading in the corresponding images. This shading was corrected based on a reference image which was obtained by smoothing the image data using a large Gaussian filter with the parameter \( \sigma = 17.4 \) mm, and where the reference image was subtracted from the original one.

![Fig. 7. Images showing the formations of the filter papers Nr 1a (top) and Nr 1b (bottom), respectively, as well as the densities of their Bartlett spectra.](image)

The function \( \hat{k}_1 \) is estimated from the image data using the method described in the previous section. The graphs of the estimates are shown in Figs. 7 to 9 (right), where \( \hat{k}_1 \) is given in mm². We remark that the relative small values of \( \hat{k}_1 \) for wave lengths \( 1/\rho \) less than 10 mm might be consequence of the shading correction. Generally, it is a hard problem to remove an unknown long wave shading under simultaneous keeping the spectrum of long waves in
the real structure. Furthermore, because of the data windowing, even the fractions of long waves are estimated with a larger error than those of short waves. Nonetheless, from studies based on realizations of GRFs with comparable spectral densities, it follows that for wave lengths less than 10 mm, the function $\hat{k}_1$ is estimated from the image data with sufficiently small errors.

Finally, a formation index $\beta$ is the determined as the mean of the density $\hat{k}_1$ for wave lengths between 2 and 5 mm, which are relevant for industrial application.

**DISCUSSION AND CONCLUSIONS**

Throughout this article it is assumed that the paper structure is isotropic, but most papers produced on papermaking machines such as those based on the principles of the Fourdrinier Machine show a clear anisotropic formation, see also Schaffnit and Dodson (1993); Scharcanski and Dodson (1996; 2000) where the anisotropy of formation is discussed in detail. Here we only remark that anisotropic paper formation corresponds to an anisotropic spectral density $\hat{k}(\xi)$, and from an estimate of $\hat{k}$ one can derive two quantities $\beta_1$ and $\beta_2$ describing the paper formation. Let $(\rho, \phi)$ denote the polar coordinates of $\xi$ with $\rho \geq 0$ and $0 \leq \phi < \pi$. Then the formation index $\beta_1$ can be computed from $\hat{k}(\rho, \phi_1)$, where $\phi_1$ is the processing direction of paper making, and $\beta_2$ is obtained from $\hat{k}(\rho, \phi_2)$ for the direction $\phi_2$ perpendicular to $\phi_1$. Usually, $\beta_1$ is at least $\beta_2$, and $\beta_1 = \beta_2$ in the isotropic case.

For fixed $\phi_1$ and $\phi_2$, the functions $\hat{k}_1^\perp(\rho) = \hat{k}(\rho, \phi_1)$ and $\hat{k}_2^\perp(\rho) = \hat{k}(\rho, \phi_2)$ can be seen as planar sections profiles of the spectral density $\hat{k}(x)$. From the projection slice theorem, see e.g. Kuba and Hermann (2008), it immediately follows that $\hat{k}_1^\perp(\rho)$ and $\hat{k}_2^\perp(\rho)$ can be obtained as a cosine transform of the orthogonal projections $k_1^\perp(r)$ resp. $k_2^\perp(r)$ of the estimated correlation function $k(x)$ onto the corresponding section planes, i.e. the rotation mean in the scheme of Fig. 6 is replaced with the orthogonal projections onto the two section planes, and the Fourier-Bessel transform is now a simple cosine transform.

As pointed out in this article, there is a 'basic formation' related to an 'independent and uniform scattering' of the paper fibers, and even this 'basic formation' can probably not be effected by technological measures. This means that the possibility to reduce paper formation by an improved paper making technology is strongly limited. Let us consider a paper with a given formation, the question is as follows: What is the difference between the given and the 'basic' formation? Unfortunately, the 'basic formation' can be estimated only roughly from the distribution of the fiber lengths and, until now there is no save method to find out whether the formation of a paper can be reduced or not.
The computation of the formation index from images of transmitted light via frequency space is very efficient. However, the results from different laboratories are comparable only under the condition that the spectral density of the intensity of the transmitted light is (nearly) the same as the spectral density of the local paper grammage. Thus, when implementing a laboratory system for industrial quality control one should take care of the wavelength of the applied light, the homogeneity of illumination, the image acquisition, a possible inhomogeneity of the paper, and the edge effects involved in the computation of the spectral density. It is very helpful to make tests as the following one: the increase of the paper weight should not influence the formation and, therefore, the paper formation of a single paper sheet must be the same as that of a double sheet (both sheets of the same formation and one sheets on top of the other). Furthermore, the estimation of the spectral density should be robust with respect to variations of the lateral resolution, i.e. varying the pixel size (in the range from 0.05 to 0.2 mm) should lead to only small changes in the estimated spectral density. Finally, the size of the paper sheet (i.e. the size of the window $W$) should be large enough such that the statistical errors of the estimates are limited. From our experience we can say that an A4-sheet is sufficient. More precisely, let $\Phi_x$ be a Gaussian random field with an exponential correlation function, $\lambda \gtrsim 0.5 \text{ mm}^{-1}$, observed through a rectangular window $W$ of the size $210 \times 297 \text{ mm}^2$, then from simulation studies it follows that the relative statistical error of an estimate of $\hat{k}_1(\rho)$ is less than 5 % for all wave lengths $1/\rho \leq 5 \text{ mm}$.

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